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On the radiation and scattering of short surface waves. Part 2

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A short-wave asymptotic analysis is undertaken for problems concerned with the radiation and scattering of surface waves by a cylinder whose cross-section S intersects the free surface normally. It is assumed that S is locally smooth and convex at the two intersection points with the fluid, which may be of infinite or finite depth. For both the scattering and radiation problem, a matched expansion technique is used to provide asymptotic estimates, in terms of relatively simple wave-free limit potentials, for the amplitudes of the surface wave trains that propagate from S. Explicit details are given for some particular geometries, confirming and extending earlier results. The method can, in principle, be extended to deal with other geometries.

1. Introduction

Problems concerned with the scattering and radiation of short surface waves by partially immersed obstacles have received the attention of many authors. The aim of the present work is to develop a relatively simple formal procedure for obtaining an asymptotic approximation for the solution throughout the whole flow field, with particular emphasis on an estimate for the amplitude of the surface waves that propagate away from the obstacle S.

It is desirable that such a procedure should provide the required information for a variety of obstacles and in a direct manner. The formal method of matched expansions, whose application to surface wave problems has been described in an earlier report (Leppington 1972), provides such a technique. Its success rests upon the reasonable assumption that the flow field can be divided into different regions in which the solution has different asymptotic forms, the argument running briefly as follows.

In the short-wave limit $\epsilon \ll a$, in which the wavelength $2\pi\epsilon$ of surface waves is small compared with a characteristic dimension a of the obstacle S, the free-surface condition can be simplified by formally letting $\epsilon \to 0$ to obtain a first approximation $\phi \sim \phi_0$ for the velocity potential (there may be a scaling factor, that depends on ϵ, multiplying ϕ_0). This *outer* approximation is assumed to provide an asymptotic estimate for ϕ at distances of many wavelengths from the free surface.

At points very close to the intersection of the obstacle S with the free surface on the other hand, a different sort of approximation is made. For in this *inner* region, within a small fraction of an obstacle dimension a from the intersection point, the potential will certainly be sensitive to the waviness of the surface, but will depend only on the *local* geometry of S. This suggests that the potential will be a slowly varying function of variables that are scaled with respect to wavelength. The formal procedure for exploiting this assumption is described in the main text, where it is shown how to calculate the amplitude of the surface waves that are formed at the outer extremities of such inner regions. Finally, the outer region is extended by continuing the surface wave trains, valid initially only within the inner regions, along the surfaces towards $x = \pm\infty$. This extension of the outer region, to include the whole surface-wave region, is discussed more fully in part 1 of this work (Leppington 1972).

With the addition of these surface waves, the outer estimate is assumed valid at distances $\gg \epsilon$ from the intersection of obstacle with the free surface, and the inner expansions are assumed valid at distances $\ll a$ from the intersection points. The two approximations therefore have a common region of validity, in which they must be asymptotically equivalent in some sense, under the assumed condition that $\epsilon \ll a$. Such a procedure inevitably involves difficulties in assigning boundary conditions to the outer and inner approximating functions. For the outer potential is not valid near the intersection points and has unknown boundary conditions there. Similarly, the inner potentials satisfy unknown conditions at infinity. Matching conditions of the type proposed by Van Dyke (1964, p. 90) provide a convenient means of exploiting the necessary equivalence of the two different approximations in their common region of validity, and complete the missing specifications for the two expansions.

The precise shape of the obstacle S near the intersection points will obviously have a crucial bearing on the inner potentials. Part 1 has dealt with geometries that are locally horizontal near their end points; this leads to inner approximations that are related to potentials involving the presence of a semi-infinite dock. Here our attention is confined to the case in which S is two-dimensional, locally convex and vertical at its ends. Thus the inner potentials involve problems of waves produced by a *plane vertical* wave maker.

The semi-circular cylinder, in a fluid of infinite depth, falls into this category, and has been dealt with rigorously by Ursell (1953) for the radiation problem and by Ursell (1961) for the scattering problem. In each case, the problem was reduced to that of an integral equation that is amenable to solution by iteration. The radiation problem has also been solved rigorously for more general geometries by Rhodes-Robinson (1970a, b, 1972), again using integral equation methods. He considers finite depth h, with any convex cylinder whose cross-section S is subject to the constraints that S intersects the surface normally, with particular emphasis on the case when S has finite radius of curvature at the ends.

Consideration is given here to the more general situation in which the cross-section S may have infinite radius of curvature at the intersection points. Thus if (x, y) axes are chosen so that the free surface is on the plane $y = 0$ with the y axis pointing into the fluid and intersection points at $(\pm a, 0)$, then S has the local form

$$x-a = -\sum_{r=N}^{\infty} \frac{\alpha_r}{r!} y^r, \quad 0 \leqslant y < y_0,$$

near $x = a$, with $2 \leqslant N < \infty$. Near the other end $(-a, 0)$ a similar expansion for $x+a$ has N replaced by M. The indices N and M give a measure of the 'steepness' of S near the points $(\pm a, 0)$, since S becomes flatter at the ends as N and M increase. As might be expected, the radiation and scattering properties of the obstacle S depend crucially on the values of these integers. The case of a plane intersecting boundary requires special treatment and is not dealt with here.

A general analysis is given, in §§ 2 and 3, for the scattering problem, in which a travelling wave train is incident upon S, and leads to a transmission coefficient of order ϵ^{N+M}, with a scale factor that depends on the precise geometry of S. Explicit details are given in § 4 for the case of infinite depth with S a semi-circle, semi-ellipse, or a semi-circle with a vertical keel. The result for the semi-circle is in agreement with that obtained rigorously by Ursell, thus giving some support to the more general predictions made. A generalization to higher order terms is briefly indicated for the semi-circle.

For the radiation problem of § 5, in which S undergoes a time-periodic heaving motion, amplitude coefficients of order ϵ^N and ϵ^M are obtained for the waves propagated towards $x = \infty$ and $x = -\infty$. These results are in agreement with those predicted by Holford (1965) and proved rigorously by Rhodes-Robinson for $N = 2$, and with those predicted by the latter author for $N = 4$.

A further generalization to finite, locally variable depth is described in § 6.

Finally, it is remarked that, although this matching procedure is not rigorous, it does provide a systematic method for solving a wide class of problems, to any order of accuracy in principle, that would be difficult to deal with by other means. For the more difficult transmission problem, Ursell's (1961) work on the semi-circular cylinder remains the only rigorous treatment; this involves finding an asymptotic estimate ϕ_0 for the potential ϕ on the body S, and then requires the very delicate task of estimating the transmission coefficient $\tilde{T}$ in terms of an integral of ϕ_0 weighted by some Green's function, evaluated over S. In spite of the rapid oscillatory nature of such integrals, Ursell finds an efficient integral formula for $\tilde{T}$, evaluates it asymptotically and shows that the error, due to higher order terms in ϕ, is negligible. The need for such a careful analysis was shown by giving another (less efficient) integral formula for $\tilde{T}$, in which the substitution $\phi \sim \phi_0$ does *not* give the right answer.

This difficult issue does not arise in the formal matching procedure. For instead of having to calculate $\tilde{T}$ in terms of the potential ϕ on S, the critical regions near the intersection points $(\pm a, 0)$ are examined directly, in great detail, by means of magnified variables scaled with respect to wavelength.

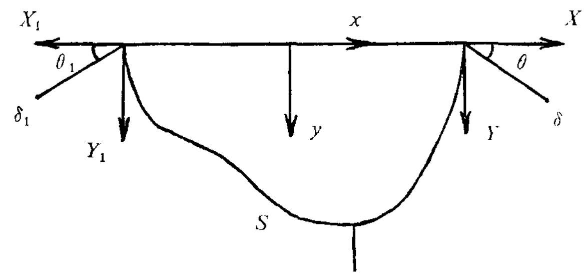

FIGURE 1. The co-ordinate systems. The curve S is vertical and locally convex at the two intersection points $(x, y) = (\pm a, 0)$.

2. Formulation of the transmission problem

A cylinder is partially immersed, with its generators horizontal, in a fluid of infinite extent. Co-ordinates are chosen so that the z direction is along the axis of the cylinder, with y pointing downward into the fluid, and the origin chosen so that the cross-sectional boundary S intersects the free surface at the points $(x, y) = (\pm a, 0)$. It is assumed that S is locally convex and has vertical tangents at these points (see figure 1).

Near $x = a$, the curve S has the form

$$x - a = -f(y), \quad y < y_0, \tag{2.1}$$

where

$$f(y) = \sum_{r=N}^{\infty} \frac{\alpha_r}{r!} y^r, \quad 2 \leqslant N < \infty, \tag{2.2}$$

with $N \geqslant 2$ to ensure the vertical tangency. If $N = 2$, then S has a finite radius of curvature $1/\alpha_2$ at the point of intersection, and S has zero curvature there if $N > 2$.

Similarly, the curve near the other point of intersection is given by

$$x + a = f_1(y), \quad y < y_0, \tag{2.3}$$

where

$$f_1(y) = \sum_{r=M}^{\infty} \frac{\beta_r}{r!} y^r, \quad 2 \leqslant M < \infty. \tag{2.4}$$

As one might expect, the reflexion and transmission properties of S depend crucially on the values of M and N; as M and N increase, the intersections become more vertical and the reflexion gets closer to being total, with less transmission.

The cylinder S is held fixed and is irradiated by an incident wave train of potential $\mathscr{R}\{\phi_i(x, y) \exp(-i\omega t)\}$, where

$$\phi_i = \exp\{-i(x-a)/\epsilon - y/\epsilon\}; \tag{2.5}$$

here ω is the angular frequency and $\epsilon = g/\omega^2$ gives a measure of the length of the travelling waves. The time factor $\exp(-i\omega t)$, which occurs throughout, will be suppressed.

The total potential $\mathscr{R}\{\phi(x, y) \exp(-i\omega t)\}$ induced by the incident wave is specified by the following linearized equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \phi = 0 \quad \text{in the fluid,} \tag{2.6}$$

$$\phi + \epsilon\phi_y = 0 \quad \text{for} \quad y = 0, |x| > a, \tag{2.7}$$

$$\partial\phi/\partial n = 0 \quad \text{on} \quad S, \tag{2.8}$$

where ϕ_y means $\partial\phi/\partial y$ and $\mathbf{n}$ is the outward normal from S. In addition we require ϕ to be finite at the ends $(\pm a, 0)$, and a radiation condition to ensure that the scattered waves travel outwards. Thus

$$\phi - \phi_i \sim \tilde{R} \exp\{i(x+a)/\epsilon - y/\epsilon\} \quad \text{as} \quad x \to \infty \tag{2.9}$$

and

$$\phi \sim \tilde{T} \exp\{-i(x-a)/\epsilon - y/\epsilon\} \quad \text{as} \quad x \to -\infty. \tag{2.10}$$

It has been shown in part 1 that, apart from additional wave-free terms, the formulae (2.9) and (2.10) hold in the extended domains $(x \mp a)/\epsilon \to \infty$. The constants $\tilde{R}$ and $\tilde{T}$ are unknowns of the problem. Our aim is to describe the potential field and, in particular, to estimate the constants $\tilde{R}$ and $\tilde{T}$, in the short-wave asymptotic limit $\epsilon \to 0$.

Following the ideas described in a related dock problem in part 1, the fluid domain is divided into overlapping regions in which different asymptotic approximations are appropriate.

The *outer region* consists of that part of the fluid that is many (small) wavelengths from the free surface. To obtain a first approximation $\phi \sim g(\epsilon)\,\phi_0$ in this region, where $g(\epsilon)$ is a scale factor to be found, we formally take $\epsilon \to 0$ in the boundary condition (2.7); the harmonic function ϕ_0 must then satisfy the homogeneous conditions

$$\left.\begin{aligned} \partial\phi_0/\partial n = 0 \quad \text{on } S; \\ \phi_0 = 0 \quad \text{for} \quad y = 0, |x| > a; \quad \phi_0 \to 0 \quad \text{at infinity.} \end{aligned}\right\} \tag{2.11}$$

Such a potential cannot carry surface waves and is clearly not valid near the free surface. In order to remedy this defect, formulae (2.9) and (2.10) show that we must add surface wave trains whose amplitudes are to be determined. For the present, it is simply noted that ϕ_0 is not valid within a few wavelengths of the surface.

It is also important to note that the homogeneous problem (2.11) for ϕ_0 must contain singularities at one or both of the ends $(x, y) = (\pm a, 0)$. Since these points lie outside the region of validity of ϕ_0, singularities are quite admissible and are to be smoothed off by solutions that are valid near the end-points.

Near to the points of intersection $(\pm a, 0)$, the waviness of the free surface cannot be ignored, but the geometry of the problem can be simplified. For if the limit $\epsilon/a \to 0$ is interpreted as keeping ϵ fixed and letting $a \to \infty$, it is clear that the solution will be scaled on the length scale ϵ and will depend only on the *local* geometry of S. Dealing with the *right inner region*, within a small fraction of a diameter $2a$ from the point $(a, 0)$, for example, this idea is formalized by rescaling the co-ordinates according to the transformation

$$x = a + \epsilon X, \quad y = \epsilon Y, \quad \phi(x, y) = \Phi(X, Y). \tag{2.12}$$

Since $x - a = -f(y)$ near the point $(a, 0)$, we have

$$\epsilon X = -\frac{\alpha_N}{N!}\epsilon^N Y^N - \frac{\alpha_{N+1}}{(N+1)!}\epsilon^{N+1} Y^{N+1} - \dots, \tag{2.13}$$

and the boundary condition (2.8) can be written as

$$\Phi_X + f'(y)\,\Phi_Y = 0, \tag{2.14}$$

when X is given by (2.13). Furthermore, the local convexity of S near the surface ensures that $\Phi(X, Y)$ is analytic on the line $X = 0,\ Y > 0$, so that Φ_X and Φ_Y may be expanded as Taylor series with respect to X; thus for $Y > 0$

$$\Phi_X(X, Y) = \Phi_X(0, Y) + X\Phi_{XX}(0, Y) + \tfrac{1}{2}X^2\Phi_{XXX}(0, Y) + \dots, \tag{2.15}$$

with a similar expansion for Φ_Y. On substituting (2.15) and (2.13) into (2.14) and expanding for small ϵ, it is found that

$$\Phi_X + \epsilon^{N-1}\frac{\alpha_N}{N!}(NY^{N-1}\Phi_Y - Y^N\Phi_{XX}) + o(\epsilon^{N-1}) = 0, \tag{2.16}$$

evaluated on the plane $X = 0$.

The problem for the right inner potential $\Phi(X, Y)$ is therefore

$$\Phi_{XX} + \Phi_{YY} = 0 \quad (X > 0, Y > 0), \tag{2.17}$$

with

$$\Phi + \Phi_Y = 0 \quad (X > 0, Y = 0), \tag{2.18}$$

together with the boundary condition (2.16) on $X = 0$.

At infinity, the formula (2.9) requires an outgoing scattered wave train at the outer extremity of this inner region, i.e.

$$\Phi \sim \exp(-iX - Y) + \tilde{R}\exp(iX - Y + 2ia/\epsilon), \tag{2.19}$$

plus a wave-free potential; the wave-free term need not be small as

$$R = (X^2 + Y^2)^{\frac{1}{2}} \to \infty,$$

since the validity of the inner potential is ensured only near $x = a$.

The precise form of the asymptotic development of the function Φ as $\epsilon \to 0$, and the final specifications regarding the behaviour of Φ at large R are to be determined by matching with the outer potential.

In the *left inner region*, within a small fraction of a diameter $2a$ from the end $(x, y) = (-a, 0)$, the co-ordinates are rescaled by the similar transformation

$$x = -a - \epsilon X_1, \quad y = \epsilon Y_1, \quad \phi(x, y) = \Psi(X_1, Y_1), \tag{2.20}$$

and the problem for the left inner potential Ψ is given by

$$(\partial^2/\partial X_1^2 + \partial^2/\partial Y_1^2)\Psi = 0 \quad (X_1 > 0, Y_1 > 0) \tag{2.21}$$

with

$$\Psi + \Psi_{Y_1} = 0 \quad (X_1 > 0, Y_1 = 0) \tag{2.22}$$

and

$$\Psi_{X_1} + \epsilon^{M-1}\frac{\beta_M}{M!}(MY_1^{M-1}\Psi_{Y_1} - Y_1^M\Psi_{X_1X_1}) + \dots = 0 \tag{2.23}$$

when $X_1 = 0$. At this side of the obstacle there are only outgoing waves, so that

$$\Psi \sim \tilde{T}\exp\{iX_1 - Y_1 + i2a/\epsilon\} \tag{2.24}$$

plus a wave-free potential, as $R_1^2 = X_1^2 + Y_1^2 \to \infty$.

According to formulae (2.9) and (2.10), the wave trains (2.19) and (2.24), valid in the first instance only within the two inner regions, continue without change of amplitude towards $x = \pm\infty$. Thus the surface wave region, many wavelengths from the ends, is completed by adding to our outer potential $g(\epsilon)\,\phi_0$ the wave trains (2.19) and (2.24). The matching of outer and inner potentials ϕ and Φ and Ψ therefore concerns only the *wave-free* parts of Φ and Ψ.

The matching procedure, and the notation used to describe this process, is the same as that described in part 1. Thus $\phi^{(m)}$ denotes the outer expansion of ϕ up to terms of order ϵ^m, and $\phi^{(m,n)}$ is the function obtained by writing $\phi^{(m)}$ in terms of the right inner variables (X, Y) and expanding for fixed (X, Y) up to terms of order ϵ^n. Likewise, $\Phi^{(n,m)}$ denotes the wave-free part of the potential $\Phi(X, Y$ expanded to order ϵ^n, written in terms of outer variables (x, y) and expanded to order ϵ^m. Our matching principle is that

$$\phi^{(m,n)} \equiv \Phi^{(n,m)} \tag{2.25}$$

for any m and n chosen at our convenience.

Matching at the other end $(-a, 0)$ is described by the notation

$$\psi^{(m,n)} \equiv \Psi^{(n,m)}, \tag{2.26}$$

where $\psi^{(m,n)}$ is the mth-order outer potential $\phi^{(m)}$ written in terms of the left inner variables (X_1, Y_1) and expanded to order ϵ^n, and similarly for $\Psi^{(n,m)}$.

A justification for the matching principle symbolized by the notation of formulae (2.25) and (2.26) is presented by Crighton & Leppington (1973).

3. First-order estimate of the transmission constant

Right inner region

The procedure described in principle in §2 is now carried out in detail to provide an asymptotic estimate for the potential in the whole fluid region. Of particular interest is an asymptotic evaluation of the transmission constant $\tilde{T}$.

Our starting point is the observation that the incident wave will be almost totally reflected as $\epsilon \to 0$, since the obstacle appears almost like a vertical wall in this limit; the boundary condition (2.16) becomes $\Phi_X = 0$ when $\epsilon = 0$. Thus for the right inner potential Φ, we tentatively pose the first approximation

$$\Phi \sim \Phi_0(X, Y) = (e^{-iX} + e^{iX})\,e^{-Y}, \tag{3.1}$$

hence

$$\tilde{R} \sim \exp(-2ia/\epsilon) \tag{3.2}$$

to this order. If a different scattered potential were tried, with a different order of magnitude from the incident wave, it would be found that it could not be matched to any outer approximation (for a similar discussion see part 1); thus (3.1) is the only possibility.

Since Φ_0 has no wave-free term at all, there is no contribution to match with an outer expansion. Now the form of the boundary condition (2.16) suggests that Φ has the development

$$\Phi \sim \Phi^{(N-1)} = \Phi_0 + \epsilon^{N-1}\Phi_1, \tag{3.3}$$

and substitution into formulae (2.16)–(2.19) leads to the following problem for the harmonic function Φ_1:

$$\Phi_1 + \Phi_{1Y} = 0 \quad (X > 0, Y = 0), \tag{3.4}$$

$$\Phi_{1X} = \frac{2\alpha_N}{N!}(NY^{N-1} - Y^N)e^{-Y} \quad (X = 0, Y > 0). \tag{3.5}$$

In addition, Φ_1 must satisfy an *outgoing* wave condition

$$\Phi_1 \sim \tilde{R}_1 \exp(iX - Y) \quad \text{as} \quad X \to \infty. \tag{3.6}$$

The possibility of additional wave-free terms, such as the eigensolution $Y-1$ satisfying the homogeneous boundary conditions, cannot be ruled out at this stage. Furthermore, the form of the expansion (3.3) needs some justification, since there might be intermediate terms of the form $g(\epsilon)\hat{\Phi}_1$, with $1 \gg g(\epsilon) \gg \epsilon^{N-1}$. Such additional functions would have a homogeneous condition in place of (3.5), and will be ruled out below on the grounds that they would be large at infinity and unmatchable to any outer expansion.

Turning now to the problem (3.4)–(3.6) for Φ_1, this is a classical wave-maker problem whose solution can be expressed as

$$\Phi_1(X, Y) = 2\int_0^\infty G(0, Y'; X, Y)\,\Phi_{1X'}(0, Y')\,dY', \tag{3.7}$$

where Φ_{1X} is given by (3.5), and G is the fundamental Green's function given by

$$2G(X', Y'; X, Y) = -2i\exp\{i|X'-X| - (Y'+Y)\} + \frac{1}{2\pi}\log\left\{\frac{(X'-X)^2+(Y'-Y)^2}{(X'-X)^2+(Y'+Y)^2}\right\}$$
$$-\frac{2}{\pi}\int_0^\infty \frac{t\cos(Y+Y')t - \sin(Y+Y')t}{1+t^2}e^{-|X'-X|t}\,dt. \tag{3.8}$$

An alternative, but equivalent form for G is given by Ursell (1961).

On substitution into (3.7) the first term in the expression for G yields the surface wave contribution

$$\Phi_1 \sim -i\alpha_N 2^{-N+1}\exp(iX - Y) \quad \text{as} \quad X \to \infty, \tag{3.9}$$

which combines with formulae (2.19) and (3.3) to show that the reflexion constant $\tilde{R}$ has the form

$$\tilde{R} \sim (1 - i\epsilon^{N-1}\alpha_N 2^{-N+1})\exp(-2ia/\epsilon). \tag{3.10}$$

In order to match our left inner approximation $\Phi \sim \Phi^{(N-1)}$ with an appropriate outer expansion $g(\epsilon)\,\phi_0$, we need some information about the form of Φ_1, at large values of $R = (X^2+Y^2)^{\frac{1}{2}}$. Setting $X = 0$, we have

$$\frac{\pi N!}{2\alpha_N}\Phi_1(Y, 0) = \int_0^\infty (NY'^{N-1} - Y'^N)e^{-Y'}dY'$$
$$\times\left\{\log\left|\frac{Y'-Y}{Y'+Y}\right| - 2\int_0^\infty \frac{t\cos(Y+Y')t - \sin(Y+Y')t}{1+t^2}dt\right\},$$

together with the exponentially decreasing wave term (3.9).

The double-integral term can be simplified by performing the Y' integral first, whence it is readily seen that its contribution to Φ_1 is of order Y^{-2}. The main contribution, of order Y^{-1}, arises from the logarithmic term, which can be recast as a Cauchy principal value integral, by partial integration, to give

$$\frac{\pi N!}{2\alpha_N}\Phi_1(0, Y) \sim 2Y \mathop{⨍}_0^\infty \frac{Y'^N}{Y^2 - Y'^2} e^{-Y'}\,dY' = 2Y^N \mathop{⨍}_0^\infty \frac{t^N}{1-t^2} e^{-Yt}\,dt,$$

with the substitution $Y' = Yt$. Finally, the use of Watson's lemma shows that

$$\Phi_1(0, Y) \sim 4\alpha_N/\pi Y \quad \text{as} \quad Y \to \infty. \tag{3.11}$$

On using the information that $\Phi_{1X}(0, Y)$ is exponentially small for large Y, it follows that the general far-field form for the harmonic function $\Phi_1(X, Y)$ is given by

$$\Phi_1(X, Y) \sim \frac{4\alpha_N}{\pi}\frac{\sin\theta}{R} \quad \text{as} \quad R \to \infty, \tag{3.12}$$

together with the surface wave (3.9), where (R, θ) are given by $X = R\cos\theta$ and $Y = R\sin\theta$.

Outer region

Leaving aside the surface wave trains, the right inner approximation

$$\Phi \sim \Phi^{(N-1)} = \Phi_0 + \epsilon^{N-1}\Phi_1$$

is now rewritten in terms of the outer co-ordinates (δ, θ), where

$$\delta = \epsilon R = \{(x-a)^2 + y^2\}^{\frac{1}{2}}$$

is the distance from the end $(a, 0)$ and is shown in figure 1. Expanding to order ϵ^N, with δ fixed, we get

$$\Phi^{(N-1,N)} = \epsilon^N \frac{4\alpha_N}{\pi}\frac{\sin\theta}{\delta}, \tag{3.13}$$

in the notation described at the end of §2. Now our matching principle

$$\Phi^{(N-1,N)} = \phi^{(N,N-1)} \tag{3.14}$$

requires the outer expansion ϕ to be such that

$$\phi \sim \phi^{(N)} = \epsilon^N \phi_0, \tag{3.15}$$

where the harmonic function ϕ_0 is to vanish at infinity and is seen from (2.11) and (3.13) to satisfy the conditions

$$\partial\phi_0/\partial n = 0 \quad \text{on} \quad S; \quad \phi_0 = 0 \quad \text{for} \quad y = 0, |x| > a, \tag{3.16}$$

and
$$\phi_0 \sim \frac{4\alpha_N}{\pi}\frac{\sin\theta}{\delta} \quad \text{as} \quad \delta \to 0. \tag{3.17}$$

We now denote by $\hat{\phi}_0$ the harmonic function that satisfies (3.16) together with the normalized singularity condition

$$\hat{\phi}_0 \sim \delta^{-1}\sin\theta \quad \text{as} \quad \delta \to 0, \tag{3.18}$$

with $\hat{\phi}_0$ *finite* at the other end $x = -a$, and $\hat{\phi}_0 = 0$ at infinity. It is clear that

$$\phi_0 = 4\alpha_N \pi^{-1} \hat{\phi}_0 \tag{3.19}$$

is a possible solution for the outer potential ϕ_0. Other eigensolutions, that could be added to (3.19), would necessarily have singularities at $x = -a$, and will be shown to be inadmissible in due course. The function $\hat{\phi}_0$ depends on the precise geometry of the scatterer S; if S is imagined to be reflected in the x axis to obtain a closed cylinder, then $\hat{\phi}_0$ represents the potential due to a dipole at $(a, 0)$, directed along the tangent to the fixed cylinder, in an unbounded fluid.

In view of the finiteness of $\hat{\phi}_0$ at $(-a, 0)$ and the boundary condition (3.16), it is readily seen that the function must take the form

$$\hat{\phi}_0 = A\delta_1 \sin\theta_1 + o(\delta_1) \quad \text{as} \quad \delta_1 \to 0, \tag{3.20}$$

near $(-a, 0)$, where $\delta_1^2 = (x+a)^2 + y^2$ and θ_1 are as shown in figure 1. The constant $A = \hat{\phi}_{0y}(-a, 0)$ obviously depends on the shape S.

Our development of the *left inner potential* and an asymptotic estimate for the transmission constant $\tilde{T}$ requires only a knowledge of the constant A that appears in formula (3.20). This constant will be evaluated for some particular geometries in § 4, and for the moment we simply assume that $\hat{\phi}_0$, and hence A, is known.

Left inner region

The outer approximation $\phi \sim \phi^{(N)} = \epsilon^N \phi_0$ is seen from (3.19) and (3.20) to have the form

$$\phi^{(N)} = \epsilon^N (4\alpha_N \pi^{-1} A\delta_1 \sin\theta_1 + o(\delta_1))$$

near the end $(-a, 0)$. When this is rewritten in left inner variables (R_1, θ_1), where $\delta_1 = \epsilon R_1$, $R_1^2 = X_1^2 + Y_1^2$, and expanded to order ϵ^{N+1}, we get

$$\psi^{(N, N+1)} = \epsilon^{N+1} A 4\alpha_N \pi^{-1} R_1 \sin\theta_1, \tag{3.21}$$

and the matching requirement $\psi^{(N, N+1)} = \Psi^{(N+1, N)}$ dictates the left inner expansion

$$\phi \sim \Psi^{(N+1)} = \epsilon^{N+1} \Psi_0. \tag{3.22}$$

According to (2.21)–(2.24) and (3.21), the function $\Psi_0(X_1, Y_1)$ is harmonic for $X_1 > 0$ and $Y_1 > 0$, and satisfies the conditions

$$\Psi_{0X_1} = 0 \quad (X_1 = 0); \quad \Psi_0 + \Psi_{0Y_1} = 0 \quad (Y_1 = 0), \tag{3.23}$$

with
$$\Psi_0 \sim A 4\pi^{-1} \alpha_N R_1 \sin\theta_1, \tag{3.24}$$

plus an outgoing wave, as $R_1 \to \infty$. Its solution is

$$\Psi_0 = A 4\pi^{-1} \alpha_N (Y_1 - 1). \tag{3.25}$$

At this stage it is possible to justify some earlier assertions that were made without proof. The fact that ϕ_0, given by (3.19), can have no singularity at $x = -a$ follows by considering the consequences of such a singularity. For the boundary conditions (3.16) satisfied by the harmonic function ϕ_0 imply the edge behaviour $\phi_0 = O(\delta_1^{\pm n} \sin n\theta_1)$, n odd; if the sign were negative, then the left inner potential would have the form $\phi \sim \epsilon^{N-n} \Psi_0^*$ in place of (3.22), and we would require the condition $\Psi_0^* \sim CR_1^{-n} \sin n\theta_1$ plus an outgoing wave as $R_1 \to \infty$, in place of (3.24). Since all inner functions have to be finite at the origin, there is no such solution apart from the trivial one $C = 0, \Psi_0^* \equiv 0$.

Similar arguments rule out the possibility of an intermediate term $g(\epsilon)\,\Phi_1^*$, in the expansion (3.3) for the right inner region, and the possibility of an additional wave-free term Φ_1^*, in the specification (3.6) for Φ_1. For in either case, the additional function would satisfy a homogeneous condition $\Phi_{1X}^* = 0$ on $X = 0$, whence $\Phi_1^* = O(R)$, or larger, at infinity; for example, $\Phi_1^* = Y - 1$ is one such term. The matching near $(a, 0)$ would therefore lead to a potential $\phi_0 = O(\delta)$, or smaller, and a non-trivial solution for ϕ_0 would have to be singular at $x = -a$. But this has been shown above to be inconsistent with respect to matching with an outgoing wave function Ψ near $x = -a$, and therefore $\Phi_1^* \equiv 0$.

Since the leading term Ψ_0, given by (3.25), is wave-free, an estimate for the transmission constant $\tilde{T}$ requires a higher order estimate for Ψ. The boundary condition (2.3) suggests a term of order ϵ^{M+N}, but we cannot rule out an intermediate term, and pose an expansion

$$\phi \sim \Psi^{(N+M)} = \epsilon^{N+1}\Psi_0 + g(\epsilon)\,\Psi_1 + \epsilon^{N+M}\Psi_2. \tag{3.26}$$

The gauge function $g(\epsilon)$ lies asymptotically between ϵ^{N+1} and ϵ^{M+N}, and is to be found. There could be several such terms, and $g(\epsilon)\Psi_1$ is taken to represent a typical term.

Each of the harmonic functions Ψ_1 and Ψ_2 satisfies the surface condition (2.22) and a radiation condition like (2.24). Substitution into (2.23) reveals the final conditions

$$\Psi_{1X_1} = 0 \quad \text{when} \quad X_1 = 0 \tag{3.27}$$

and

$$\Psi_{2X_1} = -\frac{4A}{\pi}\frac{\alpha_N \beta_M}{(M-1)!}\,Y_1^{M-1} \quad \text{when} \quad X_1 = 0. \tag{3.28}$$

The specifications for Ψ_1 and Ψ_2 are completed in principle by matching with the outer expansion to ascertain the correct behaviour for these functions at large R_1. This will not be carried out here since our main interest is with the wave trains associated with the left inner potential Ψ; it is noted here that the eigenfunction Ψ_1 is *wave-free* on account of the homogeneous condition (3.27). As for the potential Ψ_2, it is found from the identity (3.7), with X, Y and Φ_1 replaced by X_1, Y_1 and Ψ_2, that the surface train due to (3.28) is given by

$$\begin{aligned}\Psi_2 &\sim \frac{8Ai}{\pi}\frac{\alpha_N \beta_M}{(M-1)!}\,e^{iX_1 - Y_1}\int_0^\infty e^{-Y}\,Y^{M-1}\,dY \\ &= 8Ai\pi^{-1}\alpha_N \beta_M \exp\,(iX_1 - Y_1),\end{aligned} \tag{3.29}$$

for large R_1, plus a wave-free term. The formula (3.7), with Ψ_2 instead of Φ_1, is not strictly valid with Ψ_2 large at infinity, on account of convergence difficulties associated with the wave-free terms. The exact solution for Ψ_2 can readily be found, nevertheless, by subtracting out suitably chosen wave-free potentials, $g_M, g_{M-1} \ldots$, defined as

$$g_{2m} = (1/2m)\,(-1)^{m+1}\{R_1^{2m}\sin 2m\theta_1 - 2mR_1^{2m-1}\cos\,(2m-1)\,\theta_1\},$$

$$g_{2m+1} = (m+\tfrac{1}{2})^{-1}\pi^{-1}(-1)^m\{R_1^{2m+1}(\sin(2m+1)\theta_1\log R_1+\theta_1\cos(2m+1)\theta_1) \\ -(2m+1)R_1^{2m}(\cos 2m\theta_1\log R_1-\theta_1\sin 2m\theta_1)-R_1^{2m}\cos 2m\theta_1\}.$$

The functions g_n are harmonic in the quarter-space $0 \leqslant \theta_1 \leqslant \frac{1}{2}\pi$, satisfy the free-surface condition on $\theta_1 = 0$, and have been chosen so that

$$\partial g_n/\partial X_1 = Y_1^{n-1}-(n-1)\,Y_1^{n-2} \quad \text{on} \quad X_1 = 0, \quad Y_1 \geqslant 0.$$

It is easy to see that the prescribed normal velocity (3.28) on the vertical plane can be successively reduced to Y_1^{M-2}, $Y_1^{M-3}, \ldots, 1, 0$, by adding suitable multiples of the potentials $g_M, g_{M+1}, \ldots, g_2, g_1$. Since g_1 has a logarithmic singularity at the origin, this must be removed by adding an appropriate multiple of the fundamental source potential G, formula (3.8), since Ψ_2 must be finite at the origin. Thus we get a solution

$$\Psi_2 = -\frac{4A\alpha_N\beta_M}{\pi}\left\{\sum_{r=1}^{M}\frac{g_{M-r+1}}{(M-r)!}+2G(X_1,Y_1;0,0)\right\}, \tag{3.30}$$

to which may be added a combination of the eigenfunctions

$$R_1^m\sin m\theta_1 - mR_1^{m-1}\cos(m-1)\,\theta_1, \quad m \text{ odd}.$$

Reference to formula (3.8) shows that the wave train (3.29) is verified.

It has been remarked before that (2.10) ensures the continuation of the wave train (3.29) along the free surface to $x = -\infty$. Thus on being rewritten in terms of the natural variables x and y, formulae (3.29) and (3.26) show that

$$\phi \sim \tilde{T}\exp\{-i(x-a)/\epsilon - y/\epsilon\} \quad \text{as} \quad x \to -\infty,$$

where $$\tilde{T} \sim \epsilon^{N+M}8Ai\pi^{-1}\alpha_N\beta_M\exp(-2ia/\epsilon) \quad \text{as} \quad \epsilon \to 0. \tag{3.31}$$

Before providing explicit values of A for particular geometries, we note that (3.31) is consistent with the general result $\tilde{T} = O(\epsilon^4)$, predicted by Ursell (1961) for any shape S, with $N = M = 2$, that has finite radius of curvature at each end.

The symmetry of formula (3.31) with respect to M and N is also consistent with the general requirement that $\tilde{T}$ be the same for incident waves from either direction. In this connexion it is noted that the constant A, which represents the tangential velocity at $(-a, 0)$ due to a dipole at $(+a, 0)$, is also equal to the velocity at $(+a, 0)$ due to a dipole at $(-a, 0)$.

4. Special cases: circle, ellipse and circle with vertical keel

The estimate (3.31) for the transmission constant $\tilde{T}$ requires an evaluation of the constant A for a particular geometry S. Explicit results are given here for the cases of a half-submerged circular cylinder, elliptic cylinder, and circular cylinder with a vertical keel.

Circular cylinder

For a half-immersed circular cylinder the boundary S is the semi-circle

$$x^2+y^2 = a^2, \quad y > 0.$$

The harmonic function $\hat{\phi}_0$ satisfies the boundary conditions

$$\hat{\phi}_0 = 0 \quad (y = 0, |x| > a); \quad \partial\hat{\phi}_0/\partial r = 0 \quad (r = a), \tag{4.1}$$

with $$\hat{\phi}_0 \sim \delta^{-1}\sin\theta \quad \text{as} \quad \delta^2 = (x-a)^2+y^2 \to 0, \tag{4.2}$$

$\hat{\phi}_0$ finite at $(-a, 0)$ and $\hat{\phi}_0 = 0$ at infinity. The problem can be dealt with by the method of conformal transformation and has the solution

$$\hat{\phi}_0 = \delta^{-1}\sin\theta = \mathscr{R}(i/2a)(z+a)/(z-a), \tag{4.3}$$

where $z = x+iy$. Near the end $(x,y) = (-a, 0)$ we write $z = -a-\delta_1 e^{-i\theta_1}$ and expand for small δ_1 to get

$$\hat{\phi}_0 \sim (4a^2)^{-1}\delta_1\sin\theta_1, \tag{4.4}$$

and a comparison with (3.20) shows that $A = (4a^2)^{-1}$ for the semi-circle. Since S has radius a, we have $M = N = 2$ and $\alpha_2 = \beta_2 = 1/a$, and formula (3.31) shows that

$$\hat{T} \sim (\epsilon/a)^4(2i/\pi)\exp(-2ia/\epsilon). \tag{4.5}$$

This is in agreement with the result proved rigorously by Ursell (1961) (when account is made of a sign error in formula (6.1) of that work, which leads to the wrong sign for the transmitted wave).

Ellipse

If S is the semi-ellipse $(x/a)^2+(y/b)^2 = 1$, $y > 0$, then $\hat{\phi}_0$ can again be calculated exactly, using the Joukowski transformation

$$\zeta = z+(z^2-a^2+b^2)^{\frac{1}{2}} \tag{4.6}$$

to map the ellipse onto a circle of radius $a+b$; the square-root function is positive for $z > (a^2-b^2)^{\frac{1}{2}}$ and has a straight cut from $z = -(a^2-b^2)^{\frac{1}{2}}$ to $+(a^2-b^2)^{\frac{1}{2}}$. In terms of ζ, the solution is

$$\hat{\phi}_0 = \mathscr{R}(i/2b)(\zeta+a+b)/(\zeta-a-b). \tag{4.7}$$

Expanding about the point $z = -a$, $\zeta = -a-b$, leads to the conclusion that $A = (4b^2)^{-1}$; the ellipse has $M = N = 2$ with curvatures $\alpha_2 = \beta_2 = a/b^2$, whence

$$\hat{T} \sim \frac{2i}{\pi}\frac{a^2\epsilon^4}{b^6}\exp(-2ia/\epsilon). \tag{4.8}$$

Circle with vertical keel

Here S consists of the semi-circle $|z| = a$, $y > 0$, with a straight line from $z = ia$ to $z = ib$. This shape can again be reduced to a circle, of radius $(a^2+b^2)/2b$, by the transformation

$$2\zeta = \left(z^2+\frac{a^4}{z^2}+\frac{b^4+a^4}{b^2}\right)^{\frac{1}{2}}+z-\frac{a^2}{z}, \tag{4.9}$$

with cuts from ia^2/b to ib and from $-ia^2/b$ to $-ib$. The solution for $\hat{\phi}_0$ is given by

$$\hat{\phi}_0 = \mathscr{R}ib/(a^2+b^2)\{(2b\zeta+b^2+a^2)/(2b\zeta-b^2-a^2)\}, \tag{4.10}$$

from which it is found that $A = b^2/(a^2+b^2)^2$ and

$$\hat{T} \sim \epsilon^4(8i/\pi)(b/a)^2(a^2+b^2)^{-2}\exp(-2ia/\epsilon). \tag{4.11}$$

Higher order approximation for the circle

It is straightforward, in principle, to achieve higher order approximations for the circular geometry. Matching considerations require outer and inner expansions of the form

$$\phi \sim \epsilon^2\phi_0 + \epsilon^3 \log \epsilon \phi_1 + \epsilon^3 \phi_2,$$

$$\Phi \sim \epsilon\Phi_0 + \epsilon^2 \Phi_1,$$

$$\Psi \sim \epsilon^3\Psi_0 + \epsilon^4 \log \epsilon \Psi_1 + \epsilon^4\Psi_2 + \epsilon^5 \log \epsilon \Psi_3 + \epsilon^5 \Psi_4,$$

with $\Psi_0 = \pi^{-1}(R_1 \sin\theta_1 - 1)$ and Ψ_2 given as a special case of (3.30). The matching procedure yields the information that $\Psi_1 = -(4/\pi)\Psi_0$, and hence that

$$\Psi_4 = -(4/\pi)\Psi_2$$

apart from wave-free terms. It follows that $\tilde{T}$ has the improved estimate

$$\tilde{T} = \exp(-2ia/\epsilon)(2i/\pi)\{(\epsilon/a)^4 - (4/\pi)(\epsilon/a)^5 \log(\epsilon/a) + O(\epsilon/a)^5\}. \qquad (4.12)$$

5. The radiation problem

In the related radiation problem, there is no incident wave, and the motion is produced by a prescribed time-harmonic normal velocity on S. Details are given here for the case in which S undergoes a heaving motion of downward vertical velocity $\mathscr{R}(V e^{-i\omega t})$; this problem has been discussed previously by Ursell (1953), Rhodes-Robinson (1970*a*, *b*, 1972) and Holford (1965). The velocity potential $\mathscr{R}(\phi(x,y)e^{-i\omega t})$ is harmonic, finite at the intersection points $(\pm a, 0)$ and satisfies the boundary conditions

$$\phi + \epsilon\phi_y = 0 \quad \text{for} \quad y = 0, |x| > a, \qquad (5.1)$$

$$\partial\phi/\partial n = V\mathbf{j}.\mathbf{n} \quad \text{on} \quad S, \qquad (5.2)$$

$$\phi \sim A_\pm \exp(\pm ix/\epsilon - y/\epsilon) \quad \text{as} \quad x \to \pm\infty. \qquad (5.3)$$

The unit vectors $\mathbf{n}$ and $\mathbf{j}$ denote the outward normal from S, and the downward vertical direction.

In the outer region, many wavelengths from the surface, a first approximation $\phi \sim \phi_0$ is specified by setting $\epsilon = 0$ in condition (5.1). Thus

$$\phi_0 = 0 \quad \text{for} \quad y = 0, \quad |x| > a; \quad \partial\phi_0/\partial n = V\mathbf{j}.\mathbf{n} \quad \text{on} \quad S, \qquad (5.4)$$

and ϕ_0 vanishes at infinity. The addition of appropriate surface wave terms to ϕ_0 will subsequently allow the outer approximation to cover the whole fluid region except for a few wavelengths from the ends.

The two inner regions, within a fraction of a diameter $2a$ from the ends $(\pm a, 0)$, are dealt with by rescaling the variables according to formulae (2.12) and (2.20). In the right inner region, the boundary conditions (5.1) and (5.2) become

$$\Phi + \Phi_Y = 0 \quad \text{for} \quad X > 0, \quad Y = 0, \qquad (5.5)$$

$$\Phi_X + \epsilon^{N-1}\frac{\alpha_N}{N!}(NY^{N-1}\Phi_Y - Y^N\Phi_{XX}) + \dots$$
$$= \epsilon^N V \frac{\alpha_N}{(N-1)!} Y^{N-1} + \dots \quad \text{when} \quad X = 0, \qquad (5.6)$$

where the coefficient α_N is given by (2.2); the other inner region has a boundary condition obtained from (5.5) by changing X, Y, Φ, N and α_N to X_1, Y_1, Ψ, M and β_M.

Turning first to the outer approximation ϕ_0, this function, specified by (5.4), will depend on the geometry S and is assumed known for the moment: it will later be written down explicitly for the case when S is a semi-circle. Of particular importance is the behaviour of this potential near the ends $(\pm a, 0)$, for this will dictate the form of the two inner expansions. The boundary conditions (5.4) imply that ϕ_0 has the local behaviour

$$\phi_0 = AV\delta \sin\theta + o(\delta) \quad \text{as} \quad \delta \to 0 \qquad (5.7)$$

near the end $(a, 0)$, in the local co-ordinates (δ, θ) of figure 1; the constant

$$A = V^{-1}\phi_{0y}(a, 0)$$

depends on the geometry of the system and is assumed known.

Rewriting (5.7) in terms of the inner co-ordinates (R, θ) with $\delta = \epsilon R$ we have

$$\phi^{(0,1)} = AV\epsilon R \sin\theta,$$

and this suggests an inner expansion

$$\Phi \sim \epsilon\Phi_0 \qquad (5.8)$$

with
$$\Phi_0 \sim AVR\sin\theta \quad \text{as} \quad R \to \infty, \qquad (5.9)$$

together with *outgoing* waves.

On substituting (5.8) into (5.5) and (5.6), the problem for the harmonic function Φ_0 is completed by the specifications

$$\Phi_{0X} = 0 \quad (X = 0); \quad \Phi_0 + \Phi_{0Y} = 0 \quad (Y = 0), \qquad (5.10)$$

and its solution is the wave-free potential

$$\Phi_0 = AV(Y-1). \qquad (5.11)$$

Now the boundary condition (5.6) suggests a higher order expansion

$$\Phi \sim \Phi^{(N)} = \epsilon\Phi_0 + g(\epsilon)\,\Phi_1 + \epsilon^N\Phi_2, \qquad (5.12)$$

where the function $g(\epsilon)$ is to be ascertained by matching with the outer solution; there may be several such terms, all symbolized by the single intermediate expression $g(\epsilon)\Phi_1$.

Each of the constituent potentials of expansion (5.12) is harmonic and satisfies the surface condition (5.5). Substitution into the remaining condition (5.6) shows that

$$\Phi_{1X} = 0, \quad \phi_{2X} = V(1-A)\frac{\alpha_N}{(N-1)!}Y^{N-1} \qquad (5.13)$$

when $X = 0$. The eigenfunction Φ_1 is a combination of the functions

$$R^n \sin n\theta - nR^{n-1}\cos(n-1)\theta,$$

with n odd and $n \leqslant N$; it is *wave-free* and of little interest in the present analysis. The coefficients associated with the functions that make up Φ_1 would be determined by matching.

As for Φ_{2X}, it is seen from (3.28) to be proportional to the function Ψ_2 that has been described in §3. In particular then, we have

$$\Phi_2 \sim -2iV(1-A)\,\alpha_N e^{iX-Y} \quad \text{as} \quad X \to \infty. \tag{5.14}$$

Now the smooth matching of inner and outer regions is completed by continuing the wave train (5.14) to $x = \infty$. Thus from (5.12), (5.14) and (2.12) we get $\phi \sim A_+ \exp\{(ix-y)/\epsilon\}$, where

$$A_+ \sim -2i\epsilon^N V(1-A)\,\alpha_N \exp(-ia/\epsilon), \tag{5.15}$$

and a similar expression can be written down immediately for the analogous constant A_-, on replacing $\epsilon^N(1-A)\,\alpha_N \exp(-ia/\epsilon)$ by $\epsilon^M(1-A_1)\,\beta_M \exp(ia/\epsilon)$ with $A_1 = V^{-1}\phi_{0y}(-a, 0)$.

Discussion of results

The amplitude constant A_+ depends upon the value of the constant $1-A$, which gives a measure of the tangential speed of the fluid at $(a, 0)$ relative to that of S. If ϕ_0 is written as $\phi_0 = V(y-\psi)$, then ψ gives zero normal velocity on the closed cylinder S_1 made up from S and its image in the x axis. It is easy to see that ψ is the potential due to a steady stream of unit speed approaching S_1 from $y = -\infty$, and the constant $1-A$ is the slip velocity at $(a, 0)$ due to such a flow.

If S is the semi-circle of radius a, then $N = 2$ and $\alpha_N = 1/a$. The potential ϕ_0 is given exactly by

$$\phi_0 = -a^2Vy/(x^2+y^2), \tag{5.16}$$

so that $A = -1$, and formula (6.15) becomes

$$A_+ \sim -4i(V/a)\,\epsilon^2 \exp(-ia/\epsilon), \tag{5.17}$$

in agreement with the result proved rigorously by Ursell (1953).

The general result (5.15) can also be compared with those due to Rhodes-Robinson (1970*a*, *b*, 1972) for $N = 2$ and $N = 4$. If $N = 2$, so that S has finite radius of curvature at $(a, 0)$, the potential ϕ_0 is analytic near this point and can be expanded in the form

$$V^{-1}\phi_0 = \mathscr{I}\{A(z-a)+B(z-a)^2+C(z-a)^3+D(z-a)^4+\ldots\}, \tag{5.18}$$

with A, B, C and D real and $z-a = \delta e^{i\theta}$. Now the boundary condition (5.2) can be written as

$$\phi_{0x}+f'(y)\,\phi_{0y} = Vf'(y), \tag{5.19}$$

when

$$x-a = -f(y) = -\sum_2^\infty \frac{\alpha_n}{n!} y^n. \tag{5.20}$$

Substitution of the local expansion (5.18) into (5.19) yields the identity

$$V\alpha_2(1-A) = 2B = \phi_{0xy}(a,0), \tag{5.21}$$

and the general expression (5.15) can be rewritten as

$$A_+ \sim -2i\epsilon^2 \phi_{0xy}(a,0)\exp(-ia/\epsilon), \tag{5.22}$$

which is in agreement with the result obtained rigorously by Rhodes-Robinson (1970*a*, *b*, 1972) for the case of finite depth. The extension of the present analysis to finite depth is discussed in § 6.

Similarly, if $N = 4$, the expansion (5.18) can be substituted into the boundary condition (5.19) to find $B = 0$ and $V\alpha_4(1-A) = -4!\,D = +\phi_{0xyyy}(a,0)$. Thus

$$A_+ \sim -2i\epsilon^4\phi_{0xyyy}(a,0)\exp(-ia/\epsilon),$$

in agreement with the result proposed by Rhodes-Robinson (1970*a*).

The expansion (5.18) is not valid if N is odd, however. If $N = 3$, for example, then ϕ_0 has the local form,

$$V^{-1}\phi_0 = \mathscr{I}\{A(z-a)+B(z-a)^2+C(z-a)^3\log(z-a)+\ldots\},$$

with the constant A related to the coefficient C of the non-regular logarithmic term; this accounts for the difficulty described by Rhodes-Robinson (1970*a*) for the case $N = 3$. The formula (5.15) seems to provide the simplest expression for A_+ in all cases.

6. Generalization to finite depth

If the fluid is of finite depth h_1, then the scattered wave trains (2.9) and (2.10) and the radiated wave trains (5.3) have the modified wavelength parameter ϵ_1, in place of ϵ, where

$$\epsilon = \epsilon_1 \coth(h_1/\epsilon_1). \tag{6.1}$$

In addition ϕ satisfies the condition $\phi_y = 0$ when $y = h_1$. If h_1 is large compared with ϵ, then ϵ_1 is approximately $\epsilon(1-2e^{-h_1/\epsilon})$ and differs from ϵ by an exponentially small term. Thus the modification to the surface wave trains is asymptotically negligible except at very large values of x.

The modifications to the present method to deal with finite depths are very simple. The outer potentials $\hat{\phi}_0$ and ϕ_0 obviously have the boundary condition $\phi_{0y}(0,h) = 0$, in place of the previous requirement that $\phi_0 \to 0$ at infinity. Each inner potential is obtained as before; the finite depth makes its presence felt only through a modification to the scale constant A, given by (3.20) for the scattering problem and by (5.7) for the radiation problem. Finally, the surface wave trains, launched from the outer extremities of the inner regions, are continued over the surfaces towards $x = \pm\infty$, with ϵ replaced by ϵ_1, to obtain uniformity for very large values of x.

If the depth $h(x)$ is locally variable, with $h \to h_1$ as $x \to \infty$ and with $h(x) \gg \epsilon$ for all x, then the analysis will go through with little modification. The wave trains (2.10) and (5.3) have ϵ replaced by ϵ_1, where $\epsilon = \epsilon_1\coth(h_1/\epsilon_1)$, with $\tilde{T}$ and A_+ given by (3.31) and (5.15) as before. In each of these formulae the scale constant

A will account for the finite depth, through the condition $\partial\phi_0/\partial n = 0$, when $y = h(x)$, on the outer potentials ϕ_0 that determine A.

In particular, taking $N = 2$ in the radiation problem, it is asserted that the result (5.22), proved rigorously by Rhodes-Robinson (1970*a*, *b*, 1972) for finite depth, is valid for locally variable depth, for cylinders that need be smooth only within a neighbourhood of each end.

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